01/09/2025 Monday

GEOMETRIC DISTRIBUTION

Geometric distribution :- getting the first success on the xth trial in independent Bernoulli trials.

A discrete random variable X assume the values $0, 1, 2, \ldots$ with probability mass function

$$p(x) = \begin{cases} q^x \cdot p \; ; x = 0, 1, 2, \ldots \\ 0 \; ; \text{otherwise} \end{cases}$$

is called geometric distribution with parameter p. It is usually denoted by $X \sim G(p)$.

we know $\sum_{x=0}^{\infty} q^x \cdot p = 1$ and $\sum_{x=0}^{\infty} q^x = (1-q)^{-1}$

Non-central moments :-

$$\mu_1' = E(X)$$

$$= \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot q^x p$$

$$= p \sum_{x=0}^{\infty} x\, q^x$$

$$= p \sum_{x=0}^{\infty} x\, q^{x-1} q$$

$$= pq \sum_{x=0}^{\infty} x q^{x-1}$$

$$= pq\,[1 + 2q + 3q^2 + \cdots]$$

$$= pq\,[1-q]^{-2}$$

If the probability of getting a head in a coin toss is $p = 0.5$, then the probability that the 1st head appears on the 3rd toss is

$P(X=3) = (0.5)^{3-1}(0.5)^1 = (0.5)^2(0.5)$

$= 0.125$

quality control

reliability testing

$$= \frac{pq}{p^2}$$

$$\boxed{\mu_1' = \frac{q}{p}} = \text{mean.}$$

$$\mu_2' = E(X^2) = E[X(X-1)+X]$$

$$= E[X(X-1)] + E(X)$$

$$= \sum_{x=0}^{\infty} x(x-1)\, p(x) + \frac{q}{p} = \sum_{x=0}^{\infty} x(x-1)\, q^x p + \frac{q}{p}$$

$$= p\sum_{x=0}^{\infty} x(x-1)\, q^{x-2}\, q^2 + \frac{q}{p}$$

$$= pq^2 \sum_{x=0}^{\infty} x(x-1)\, q^{x-2} + \frac{q}{p}$$

$$= 2!\, pq^2 \sum_{x=2}^{\infty} \frac{x(x-1)}{2!}\, q^{x-2} + \frac{q}{p}$$

$$= 2!\, pq^2 (1-q)^{-3} + \frac{q}{p}$$

$$= 2!\, pq^2 p^{-3} + \frac{q}{p}$$

$$\boxed{\mu_2' = \frac{2q^2}{p^2} + \frac{q}{p}}$$

$$\text{var} = \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2}$$

$$= \frac{q}{p}\left[\frac{2q}{p} + 1 - \frac{q}{p}\right]$$

$$= \frac{q}{p}\left[\frac{2q+p-q}{p}\right] = \frac{q}{p}\left[\frac{q+p}{p}\right] = \frac{q}{p^2}$$

**Moment generating function of Geometric distribution :-

$$M_X(t) = E(e^{tx})$$
$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$
$$= \sum_{x=0}^{\infty} e^{tx} p q^x$$
$$= p \sum_{x=0}^{\infty} (qe^t)^x$$
$$\boxed{M_X(t) = p(1-qe^t)^{-1}}$$ or
$$\boxed{M_X(t) = \frac{p}{(1-qe^t)}}$$ //

* C.F of geometric distribution :-

$$\phi_X(t) = E(e^{itx})$$
$$= \sum_{x=0}^{\infty} e^{itx} q^x \cdot p$$
$$= p \sum_{x=0}^{\infty} e^{itx} q^x$$
$$= p \sum_{x=0}^{\infty} (e^{it} q)^x$$
$$\boxed{\phi_X(t) = p(1-qe^{it})^{-1}}$$ //

* P.G.F of Geometric distribution :-

$$P_X(s) = E(s^x)$$
$$= \sum_{x=0}^{\infty} s^x q^x \, p$$

$$= p \sum_{x=0} s^x q^x$$

$$= p \sum_{x=0}^{\infty} (sq)^x$$

$$\boxed{P_x(s) = p(1-qs)^{-1}}$$

* C.G.F of Geometric distribution :-

$$K_X(t) = \log M_X(t) = \log p(1-qe^t)^{-1}.$$

$$= \log p - \log(1-qe^t)$$

$$= \log p - \log\left[1 - q\left(1+t+\frac{t^2}{2!}+\cdots\right)\right]$$

$$= \log p + \left\{ q\left(1+t+\frac{t^2}{2!}+\cdots\right) + \frac{\left[q\left(1+t+\frac{t^2}{2!}+\cdots\right)\right]^2}{2} + \cdots \right.$$

Cummulants

$$\text{Mean} = K_1 = \boxed{\mu_1' = \frac{q}{p}}$$

$$\text{Var} = K_2 = \boxed{\mu_2 = \frac{q}{p^2}}$$

$$K_3 = \boxed{\mu_3 = \frac{q(1+q)}{p^3}}$$

$$\mu_4 = K_4 + 3K_2^2 = \frac{q}{p^2}\left(1+\frac{6q}{p^2}\right) + 3\left(\frac{q}{p^2}\right)^2$$

$$\boxed{\mu_4 = \frac{q(p^2+9q)}{p^4}}$$

Imp

* Mean and variance through M.G.F :-

$$\mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} \qquad \mu_r' = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt} p(1-qe^t)^{-1}\right]_{t=0}$$

$$= \left[p(-1)(1-qe^t)^{-2}(-qe^t)\right]_{t=0}$$

$$= p(1-qe^0)^{-2}.q$$

$$= pp^{-2}.q$$

$$\boxed{\mu_1' = \frac{q}{p}}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d^2}{dt^2} p(1-qe^t)^{-1}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\cdot\frac{d}{dt} p(1-qe^t)^{-1}\right]_{t=0}$$

$$= \left[\frac{d}{dt} pq(1-qe^t)^{-2}.e^t\right]_{t=0}$$

$$= pq\left[(1-qe^t)^{-2}e^t + e^t(-2)(1-qe^t)^{-3} * (-qe^t)\right]_{t=0}$$

$$= pq\left[(1-q)^{-2} + 2(1-q)^{-3}.q\right]$$

$$= pq\left[\frac{1}{p^2} + \frac{2q}{p^3}\right]$$

$$\boxed{\mu_2' = \frac{q}{p} + \frac{2q^2}{p^2}}$$

$$\text{Var } \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q}{p}\left[\frac{2p}{q} + 1 - \frac{q}{p}\right]$$

$$= \frac{q}{p}\left[\frac{q+p}{p}\right] = \frac{q}{p^2}.$$

$$\boxed{\mu_2 = \frac{q}{p^2}}$$

* Skewness of Geometric distribution:-

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\left[\frac{q(1+q)}{p^3}\right]^2}{\left(\frac{q}{p^2}\right)^3}.$$

$$\boxed{\beta_1 = \frac{(1+q)^2}{q}}$$

$$\boxed{\gamma_1 = \sqrt{\beta_1} = \frac{1+q}{\sqrt{q}}}$$

* Kurtosis of Geometric distribution:-

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\frac{q(p^2+9q)}{p^4}}{\left(\frac{q}{p^2}\right)^2} = \frac{p^2+9q}{q}$$

$$\boxed{\beta_2 = \frac{p^2+9q}{q}}$$

$$\gamma_2 = \beta_2 - 3 = \frac{p^2+6q}{q}$$

$$\boxed{\gamma_2 = \frac{p^2+6q}{q}}$$

* Recurrence relation for probabilities of geometric distribution:-

$p(x) = q^x \cdot p$; $x = 0, 1, 2, \ldots$

$p(x+1) = q^{x+1} \cdot p$

$$\frac{p(x+1)}{p(x)} = \frac{q^{x+1}}{q^x \cdot p} = q$$

$$p(x+1) = q\, p(x).$$

* Additive property of geometric distribution:-

statement:- Sum of n independent geometric variates is also a geometric negative binomial variates.

proof:- Let $x_1, x_2, \ldots x_n$ be independent geometric variables with the parameter of p.

The M.G.F's are $M_{x_1}(t) = p(1-qe^t)^{-1}$ + $M_{x_2}(t)$ + ... $M_{x_n}(t) = p(1-qe^t)^{-1}$

$$M_{\sum_{i=1}^{n} x_i}(t) = M_{x_1+x_2+\cdots x_n}(t)$$

$$= M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t)$$

$$= p(1-qe^t)^{-1} \cdot p(1-qe^t)^{-1} \cdots p(1-qe^t)^{-1}$$

$$= p^n (1-qe^t)^{-n}$$

which is M.G.F of geometric negative binomial distribution with parameter's (n, p)

* lack of memory property of geometric distribution:-

Statement :- X follows a geometric distribution for $K \geq 0$, $y = x - K$ as the geometric distribution, called lack's memory.

i.e. $p(y \geq t / x \geq K) = p(x \geq t)$

proof: $p(x \geq t) = q^t . p$, $t = 0, 1, 2, \ldots$

$p(y = t / x \geq K) = p(y \geq t / x \geq K) - p(y \geq t+1 / x \geq K)$ — ①

consider

$$p(y \geq t / x \geq K) = \frac{p(y \geq t \cap x \geq K)}{p(x \geq K)}$$

$$= \frac{p(x - K \geq t \cap x \geq K)}{p(x \geq K)}$$

$$= \frac{p(x \geq K + t \cap x \geq K)}{p(x \geq K)}$$

$$= \frac{p(x \geq K+t)}{p(x \geq K)} \quad \text{— ②}$$

consider $p(x \geq K) = \sum_{x=K}^{\infty} x\, p(x) = \sum_{x=K}^{\infty} q^t . p$

$= p[q^k + q^{k+1} + \dots]$

$= pq^k[1+q+q^2+\dots]$

$= pq^k(1-q)^{-1}$

$= pq^k \cdot p^{-1}$

$= q^k$

Similarly,

$p(X \geq k+t) = q^{k+t}$

from ②

$$p(Y \geq t / X \geq k) = \frac{q^{t+k}}{q^k} = q^t$$

and hence,

$$p(Y \geq t+1 / X \geq k) = q^{t+1}$$

from eq ①

$$p(Y=t / X \geq k) = q^t - q^{t+1} = \frac{p(X \geq k+t)}{p(X \geq k)} \quad -②$$

$-p(Y \geq t+1 / X \geq k)$

$= q^t(1-q)$

$= q^t p$

$= p(X=t)$

Hence, the result.

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